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OPTIMAL DESIGN OF ELASTIC STRUCTURES FOR MAXIMUM STIFFNESS†

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Abstract—The purpose of this paper is to establish a general theory of optimal design of elastic structures such that the structure with a given volume would have maximum stiffness. A sufficient condition of optimality is derived from the principle of minimum potential energy. This optimality condition is proven by the variational method to be a necessary one under the condition that the optimal structure has certain continuity and differentiability properties. Physical interpretations of the optimality condition are discussed for problems of beams, plates and trusses. An application of the theory is illustrated in a problem of optimal design of a simply supported circular plate under uniform pressure. Detailed description of the numerical procedure for the solution of the plate problem is presented.

1. INTRODUCTION

IN the problem of optimal elastic design of structures for maximum stiffness,§ we usually define a design variable and ask how the design variable should vary as a function of position such that the work done by a given load to the structure is maximized while the total weight of the structure remains constant. The choice of the design variable depends entirely on the design purpose. For example, in the problem of optimal design of solid beams, we may choose the height of the beam as a design variable and keep the width of the beam fixed or vice versa. In the problems of optimal design of plates we would consider the thickness of the plate as a design variable. Equivalent problems of optimal design are: (i), for given deformation and load, find the variation of the design variable such that the total weight of a structure is a minimum; and (ii), for given deformation and total weight of a structure, find the variation of the design variable such that the applied load is a maximum.

A general method of treating a variety of problems of optimal design of sandwich structures was given by Prager and Taylor [1]. The problems of optimal elastic design of solid beams to achieve the highest fundamental frequency and of solid columns to reach the largest buckling load or maximum height have been analyzed by the variational method in [2–4].

In this paper, we shall deal with the optimal design of elastic structures for maximum stiffness in a general manner. A necessary and sufficient condition of optimality is derived by means of the calculus of variation and the principle of minimum potential energy. The physical meanings of this optimality condition in the problems of beams, plates and trusses are discussed. An application of the theory is made for the problem of optimal design of a circular plate with simply supported edge under a uniform pressure.

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§ The term "stiffness" used in this paper is measured by the external work done by the load.

2. BASIC PRINCIPLE

Consider an elastic structure which may be a beam, a plate, a shell or a truss. Let P_i and δ_i be the generalized load and the generalized displacement at any point. The external work is

$$W = \tfrac{1}{2}\sum_i P_i\delta_i = \int U\,\mathrm{d}S \tag{1}$$

where dS denotes the line element of a one-dimensional structure or the area element of a two-dimensional structure and U is the unit strain energy. The potential energy is defined as

$$\Pi = \int U\,\mathrm{d}S - \sum_i P_i\delta_i. \tag{2}$$

Therefore

$$\Pi = -W. \tag{3}$$

Let s be the stiffness and e the specific strain energy or the strain energy per unit stiffness, we have

$$U = se \tag{4}$$

Denote the design variable by $B = B(S)$ which is a function of position. For an appropriate choice of the design variable, the total volume of the structure can be written as

$$V = c\int B\,\mathrm{d}S \tag{5}$$

where c is a positive constant. The stiffness can usually be expressed as

$$s = \alpha B^n \tag{6}$$

where α is a positive constant and n stands for any positive integer. For example, in the problem of optimal design of a beam of rectangular cross section with fixed width b and variable height for maximum bending stiffness, we may take the cross-sectional area of the beam as the design variable. We then have $c = 1$, $n = 3$ and $\alpha = E/(12b^2)$, where E is Young's modulus for the beam material.

In general, we have from equations (2), (3), (4) and (6),

$$\Pi = -W = \alpha\int eB^n\,\mathrm{d}S - \sum_i P_i\delta_i. \tag{7}$$

For given load, we can consider W as a measure of stiffness.

Let us consider two designs specified by $B(S)$ and $\bar{B}(S)$. Suppose that these two designs would lead to the same stiffness, i.e.,

$$\alpha\int eB^n\,\mathrm{d}S - \sum_i P_i\delta_i = \alpha\int \bar{e}\bar{B}^n\,\mathrm{d}S - \sum_i P_i\bar{\delta}_i \tag{8}$$

where $\bar{e}$ and $\bar{\delta}_i$ are respectively the specific strain energy and generalized displacement corresponding to the design $\bar{B}$. By the principle of minimum potential energy, we have

$$\alpha\int \bar{e}\bar{B}^n\,\mathrm{d}S - \sum_i P_i\bar{\delta}_i \le \alpha\int e\bar{B}^n\,\mathrm{d}S - \sum_i P_i\delta_i \tag{9}$$

because the displacements of the design B are also kinematically admissible for the design $\bar{B}$. From equation (8) and the inequality (9), we have

$$\int e(\bar{B}^n - B^n)\,\mathrm{d}S \geq 0. \tag{10}$$

If $\bar{B}$ is infintesimally close to B, we may set

$$\bar{B} = B + \delta B \tag{11}$$

with δB to be a function of position of infinitesimal magnitude. If we substitute equation (11) into the inequality (10) and neglect the higher order terms of δB we obtain

$$\int eB^{n-1}\delta B\,\mathrm{d}S \geqslant 0. \tag{12}$$

If

$$eB^{n-1} = \text{constant} \tag{13}$$

then $\int \delta B\,\mathrm{d}S \geq 0$ and the design specified by B is an optimal one in the sense that it does not have a greater structural weight than any neighboring design of the same stiffness. Thus equation (13) is a sufficient condition for optimality in this restricted sense. Equation (13) can also be written as

$$U/B = \text{constant}. \tag{14}$$

We shall prove that the optimality condition, equation (13), is also a necessary condition of optimal design. For given V, the problem of optimal design for maximum stiffness can be reduced to find $B(S)$ such that the external work W given by equation (1) is maximized under the subsidiary condition of equation (5). If the design variable is continuous and differentiable, this problem can be treated by the calculus of variation. We can set the following variational equation from equations (1), (4), (5) and (6):

$$\delta \int (eB^n - \lambda cB)\,\mathrm{d}S = 0, \tag{15}$$

where λ is a Lagrange multiplier. The Euler equation of equation (15) would lead to the optimality condition equation (13). Thus equation (13) is also a necessary condition of optimal design.

By the condition of optimality and the equation of equilibrium, we can find a solution for our problem corresponding to a local minimum weight of the structure if this solution exists. Whether this local minimum is also the absolute minimum depends on the uniqueness of the solution.

In the problems of optimal design of beams, it is evident from equation (14) that a beam design is optimal in the sense specified above if the given loads produce deflections such that the strain energy in any element between adjacent cross sections is proportional to the weight of this element. Equation (14) can also be written as

$$M^2/(IA) = \text{constant} \tag{16}$$

where M is the bending moment and A and I are the area and the moment of inertia of the cross section respectively. We can also write equation (16) as

$$Mh/(IAh^2)^{\frac{1}{2}} = \text{constant}. \tag{17}$$

where h is the height of beams of rectangular cross section or the radius of beams of circular cross section. Since I is proportional to Ah^2, we have

$$Mh/I = \text{constant}. \tag{18}$$

Accordingly, the optimal design of beams of rectangular or circular cross section is also the design of uniform strength in which the stresses in the extreme fibres are uniform. By using this property of uniform strength, we can simplify our analysis of optimal design of beams, particularly when the problem is statically determinate.

In the problem of optimal design of plates under bending and without stretching, if the thickness of the plate h is taken as the design variable, equation (14) becomes

$$[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2]/h^4 = \text{constant} \tag{19}$$

where M_x and M_y are bending moments, M_{xy} is the twisting moment and ν is the Poisson's ratio of the material. Equation (19) can also be written as

$$\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2 = \text{constant} \tag{20}$$

where σ_x, σ_y and τ_{xy} are stress components on the surfaces of the plate. An alternative expression of equation (20) is

$$\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy} = \text{constant} \tag{21}$$

where ε_x, ε_y and γ_{xy} are strain components on the surfaces of the plate. Therefore the optimal design of plates under bending and without stretching is the design of constant strain energy density on the surfaces of the plate. This condition of constant strain energy density on the surfaces is of course also applicable to the beam problems.

We can also prove by the same method as shown in the beam and plate problems that the optimum truss of given configuration and volume for maximum stiffness is the one in which the stresses per unit area in all members have the same magnitude. In other words, the optimal design of trusses is also the design of uniform strength.

3. OPTIMAL DESIGN OF CIRCULAR PLATES WITH SIMPLY SUPPORTED EDGE UNDER UNIFORM PRESSURE

As an illustrative example of the theory developed in the previous section, let us consider the problem of optimal design of an elastic circular plate of radius a under a uniform pressure q for maximum stiffness. Again, we shall measure the stiffness of the plate by the external work of pressure q. Set the origin of a polar coordinates system at the center of the plate. We shall assume that the optimum plate is axisymmetric, i.e., the thickness h and deflection w of the plate are functions of radial coordinate r only. The curvatures of the deformed plate are

$$\varkappa_r = -w'' \tag{22}$$

and

$$\varkappa_\theta = -\frac{1}{r}w' \tag{23}$$

where $(\ )' = d/dr(\ )$. The moment–curvature relations are

$$M_r = D(\varkappa_r + \nu\varkappa_\theta) = -D\left(w'' + \frac{\nu}{r}w'\right) \tag{24}$$

and

$$M_\theta = D(\varkappa_\theta + \nu\varkappa_r) = -D\left(\frac{1}{r}w' + \nu w''\right) \tag{25}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{26}$$

The equation of equilibrium is

$$\frac{M_r}{r} + M_r' - \frac{M_\theta}{r} + Q = 0 \tag{27}$$

where

$$Q = qr/2 \tag{28}$$

is the transverse shear. Substitution of equations (24), (25) and (28) into equation (27) leads to

$$D(\nabla^2 w)' + D'\left(w'' + \frac{\nu}{r}w'\right) = \tfrac{1}{2}qr \tag{29}$$

where $\nabla^2 = (d^2/dr^2) + (1/r)(d/dr)$ is the Laplace operator. Equation (29) can be written as

$$2\alpha h^3\frac{1}{r}(\nabla^2 w)^2 + 6\alpha h^2 h'\frac{1}{r}\left(w'' + \frac{\nu}{r}w'\right) = q \tag{30}$$

where $\alpha = E(1-\nu^2)^{-1}/12$. If we choose the thickness of the plate h as the design variable, the condition of optimality, equation (14), becomes

$$U/h = \tfrac{1}{2}\alpha h^2\left[(\nabla^2 w)^2 - 2(1-\nu)\frac{1}{r}w'w''\right] = C \tag{31}$$

where C is a constant.

Let $\bar{w}$ be the average deflection of the plate. We have

$$\bar{w} = \frac{2}{a^2}\int_0^a wr\,dr. \tag{32}$$

From equation (1), we have

$$W = \tfrac{1}{2}\pi q\bar{w}a^2 = 2\pi\int_0^a Ur\,dr = 2\pi C\int_0^a hr\,dr = CV. \tag{33}$$

After elimination of C from equations (31) and (33), we obtain

$$2\alpha vh^2\left[(\nabla^2 w)^2 - 2(1-\nu)\frac{1}{r}w'w''\right] / (\bar{w}a^2) = q \tag{34}$$

where

$$v = \frac{V}{2\pi} = \int_0^a hr\,\mathrm{d}r. \tag{35}$$

The edge of the plate is assumed to be simply supported. We have the following boundary conditions:

$$w'(0) = 0, \tag{36}$$

$$w(a) = 0, \tag{37}$$

and

$$M_r(a) = 0. \tag{38}$$

Substituting equation (24) into equation (38) and using equation (34), we have

$$\left(w'' + \frac{v}{r}w'\right)\left[(w'')^2 + 2v\frac{1}{r}w'w'' + \left(\frac{w'}{r}\right)^2\right]^{-\frac{1}{2}} = 0 \quad \text{at} \quad r = a. \tag{39}$$

Since $w'(a)$ is bounded,

$$w''(a) = \infty. \tag{40}$$

From equations (24), (38) and (40), we have

$$h(a) = 0. \tag{41}$$

Let us introduce the following dimensionless quanti ies

$$x = r/a, \qquad y = 2\chi v^3 w/(qa^{10}) \quad \text{and} \quad t = a^2 h/v \tag{42}$$

and put

$$(\quad)' = \frac{\mathrm{d}}{\mathrm{d}x}(\quad) \quad \text{and} \quad \nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Equations (35), (30), (34), (32), (36), (37), (40) and (41) can be expressed in terms of these dimensionless quantities:

$$\int_0^1 tx\,\mathrm{d}x = 1, \tag{43}$$

$$t^3\frac{1}{x}(\nabla^2 y)' + 3t^2 t'\frac{1}{x}\left(y'' + \frac{v}{x}y'\right) = 1, \tag{44}$$

$$t^2\left[(\nabla^2 y)^2 - 2(1-v)\frac{1}{x}y'y''\right] = \bar{y}, \tag{45}$$

$$\bar{y} = 2\int_0^1 yx\,\mathrm{d}x, \tag{46}$$

$$y'(0) = 0, \tag{47}$$

$$y(1) = 0, \tag{48}$$

$$y''(1) = \infty \tag{49}$$

and

$$t(1) = 0. \tag{50}$$

Let $\theta = y'$ be the deflection slope. Equations (44), (45), (47), and (49) can be written in terms of θ as

$$t^3\frac{1}{x}\left(\theta''+\frac{1}{x}\theta'-\frac{\theta}{x^2}\right)+3t^2t'\frac{1}{x}\left(\theta'+\frac{v}{x}\theta\right) = 1, \tag{51}$$

$$t^2\left[\theta'^2+2v\frac{1}{x}\theta\theta'+\frac{1}{x^2}\theta^2\right] = 2\int_0^1 x\int_1^x \theta(\zeta)\,d\zeta\,dx, \tag{52}$$

$$\theta(0) = 0 \tag{53}$$

and

$$\theta'(1) = \infty. \tag{54}$$

Equations (43) and (50)–(54) are the governing equations of our problem. Since the non-linearity is involved in the coupled differential equations, we shall solve them by an iterative procedure. First of all, we shall assume values for $\theta(x)$, then solve for $t(x)$ from equation (51) and adjust $\theta(x)$ such that equation (43) is satisfied. Next, $\theta(x)$ can be solved from equations (52), (53) and (54) and $t(x)$ is solved from equations (43) and (51) again. Such iteration will continue until certain convergence criterion is fulfilled. The process involved in this iterative procedure will be described in detail in the following.

From equation (54) we can see that close to the edge of the plate there is a region in which $\theta'' \gg \theta' \gg \theta$ and equations (51) and (52) can be expressed approximately as

$$t^3\frac{1}{x}\theta''+3t^2t'\frac{1}{x}\theta' = 1 \tag{55}$$

and

$$t^2\theta'^2 = 2\int_0^1 x\int_1^x \theta(\zeta)\,d\zeta\,dx = 4c_1^4 \tag{56}$$

where c_1 is a constant. Equations (55), (56) and (50) can be solved explicitly. The solutions are

$$\theta = -4c_1^3\sin^{-1}x+c_2 \tag{57}$$

and

$$t = \frac{1}{2c_1}(1-x^2)^{\frac{1}{2}} \tag{58}$$

where c_2 is an integration constant. Let $1-x^*$ be the thickness of the boundary layer region in which the approximate equations (55) and (56) hold. The values of c_1 and c_2 are determined by matching the values of θ and θ' at $x = x^*$, viz.,

$$-4c_1^3 = \theta'(x^*)(1-x^{*2})^{\frac{1}{2}} \tag{59}$$

and

$$c_2 = \theta(x^*)+4c_1^3\sin^{-1}x^*. \tag{60}$$

Let $x = x_i = (i-1)\Delta$, where Δ is the mesh size. Set $\theta_i = \theta(x_i)$ and $t_i = t(x_i)$. In the region $x < x^*$, we shall use the following central difference equations to approximate the derivatives of $\theta(x)$:

$$\theta_i' = \frac{1}{2\Delta}(\theta_{i+1} - \theta_{i-1}) \tag{61}$$

and

$$\theta_i'' = \frac{1}{\Delta^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}). \tag{62}$$

Put

$$\varphi(x) = \frac{1}{x}\left(\theta'' + \frac{1}{x}\theta' - \frac{1}{x^2}\theta\right) \tag{63}$$

and

$$\psi(x) = \frac{3}{x}\left(\theta' + \frac{\nu}{x}\theta\right). \tag{64}$$

Equation (51) can be written as

$$t' = \frac{1}{\psi t^2}(1 - \varphi t^3). \tag{65}$$

We shall use the open Adams method

$$t_i = t_{i-1} + \frac{\Delta}{2}(t_{i-2}' - 3t_{i-1}') \tag{66}$$

to calculate t_i in the first iteration and use the modified Euler's method

$$t_i = t_{i-1} - \frac{\Delta}{2}(t_i' + t_{i-1}') \tag{67}$$

to evaluate t_i in the rest of the iterations. The values of the calculated $t(x)$ depend on the values of $\theta(x)$ used in equations (61–64). The values of $t(x)$ thus calculated may not satisfy equation (43). Let

$$\int_0^1 tx\,\mathrm{d}x = \mu. \tag{68}$$

Then the adjusted values of $t(x)$ and $\theta(x)$ would be

$$t_a(x) = t(x)/\mu \tag{69}$$

and

$$\theta_a(x) = \mu^3\theta(x). \tag{70}$$

Note that such adjustments would cause $t_a(x)$ and $\theta_a(x)$ to satisfy equations (43) and (51).

After $t(x)$ is determined we shall solve for $\theta(x)$ from equations (52), (53) and (54). Let us define $\tilde{\theta}(x)$ such that it satisfies the following equations:

$$t^2\left(\tilde{\theta}'^2+2\nu\frac{1}{x}\tilde{\theta}\tilde{\theta}'+\frac{1}{x^2}\tilde{\theta}^2\right) = 1 \tag{71}$$

$$\tilde{\theta}(0) = 0 \tag{72}$$

and

$$\tilde{\theta}'(1) = \infty. \tag{73}$$

For given $t(x)$ we shall solve for $\tilde{\theta}(x)$. Let

$$\beta(x) = x^2/t^2. \tag{74}$$

Equation (71) can be written as

$$x^2\tilde{\theta}'^2+2\nu x\tilde{\theta}\tilde{\theta}'+\tilde{\theta}^2 = \beta. \tag{75}$$

Therefore

$$\tilde{\theta}' = -\frac{1}{x}\{\nu\tilde{\theta}\pm[\beta-(1-\nu^2)\tilde{\theta}^2]^{\frac{1}{2}}\}. \tag{76}$$

Since

$$\lim_{x\to 0}\frac{\tilde{\theta}}{x} = \tilde{\theta}'(0), \tag{77}$$

from equation (71), we have

$$2(1+\nu)[t(0)\tilde{\theta}'(0)]^2 = 1 \tag{78}$$

or

$$\tilde{\theta}'(0) = -[2(1+\nu)]^{-\frac{1}{2}}[t(0)]^{-1}. \tag{79}$$

If we consider equation (79) as a limiting case of equation (78), we have

$$\tilde{\theta}' = -\frac{1}{x}\{\nu\tilde{\theta}+[\beta-(1-\nu^2)\tilde{\theta}^2]^{\frac{1}{2}}\}. \tag{80}$$

Let $\tilde{\theta}_i = \tilde{\theta}(x_i)$, we have

$$\tilde{\theta}_1 = 0. \tag{81}$$

$$\tilde{\theta}'_1 = -[2(1+\nu)]^{-\frac{1}{2}}t_1^{-1}. \tag{82}$$

At the beginning, we shall try

$$\tilde{\theta}_2 = -\Delta[2(1+\nu)]^{-\frac{1}{2}}t_1^{-1}, \tag{83}$$

then evaluate $\tilde{\theta}'_2$ from equation (80) and $\tilde{\theta}_2$ by iterations according to the following quadrature formula:

$$\tilde{\theta}_2 = \frac{\Delta}{2}(\tilde{\theta}'_1+\tilde{\theta}'_2). \tag{84}$$

After $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are evaluated, we shall calculate $\theta_i (i > 2)$ in the region $x \leq x^*$ according to the following procedure:

(i) To use the open Adams method for the first iteration, i.e.,

$$\tilde{\theta}_{i+1} = \tilde{\theta}_i + \frac{\Delta}{2}(3\tilde{\theta}'_i - \tilde{\theta}'_{i-1}). \tag{85}$$

(ii) To use the modified Euler's method for the rest of the iterations, i.e.,

$$\tilde{\theta}_{i+1} = \tilde{\theta}_i + \frac{\Delta}{2}(\tilde{\theta}'_i + \tilde{\theta}'_{i+1}). \tag{86}$$

We shall choose x^* such that it is a grid point. After $\tilde{\theta}(x^*)$ is calculated, we shall evaluate c_1 and c_2 according to equations (59) and (60) by replacing $\theta(x^*)$ by $\tilde{\theta}(x^*)$ and $\theta'(x^*)$ by $\tilde{\theta}'(x^*)$. For $x > x^*$, $\tilde{\theta}(x)$ is determined according to equation (57).

Let

$$\omega = 2\int_0^1 x \int_1^x \tilde{\theta}(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x. \tag{87}$$

If we set

$$\theta(x) = \omega\tilde{\theta}(x), \tag{88}$$

then

$$2\int_0^1 x \int_1^x \theta(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x = \omega^2 \tag{89}$$

and $\theta(x)$ would satisfy equations (52), (53) and (54).

This iterative procedure will continue until the following convergence criterion is fulfilled:

$$\max_x |[\theta(x) - \theta^p(x)]/\theta^p(x)| < 0.2\% \tag{90}$$

where $\theta^p(x)$ is the value of $\theta(x)$ appeared in the previous process of iteration. At the beginning of our calculation, we shall take

$$\theta(x) = -\sin^{-1} x \tag{91}$$

as the initial tried value of $\theta(x)$.

The numerical values of $t(x)$ and $\theta(x)$ are shown in Fig. 1 for $\nu = \frac{1}{3}$. The average deflection of the optimum plate is found to be

$$\bar{w} = \frac{\omega^2}{2}\frac{qa^{10}}{\chi v^3} = 0.5613\frac{qa^{10}}{\chi v^3}. \tag{92}$$

For a plate of the same volume and uniform thickness, the average deflection is

$$\bar{w} = \frac{11}{384}\pi^3\frac{qa^{10}}{\chi v^3}. \tag{93}$$

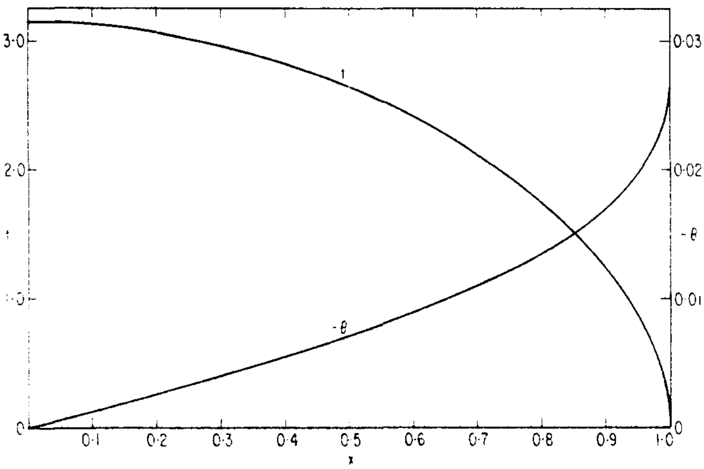

FIG. 1. $t(x)$ and $\theta(x)$ curves for $\nu = 1/3$.

Therefore the ratio of the stiffness of the optimum plate to that of a plate of the same volume and uniform thickness is

$$r_s = 1{\cdot}582. \tag{94}$$

4. DISCUSSION

The proof that the optimality condition, equation (13), is the necessary condition of optimal design is based on the assumption of continuity and differentiability of the design variation function. Megarefs [5] has shown in a problem of plastic minimum weight design or circular solid plate for maximum safety that the volume of the plate can be reduced if one allows the discontinuity in the variation of plate thickness. In view of the use of piecewise linear yield conditions with an infinity of compatible strain rates at vertices or edges of the yield locus, compatibility is less stringent in plasticity than in elasticity. Certain conditions of continuity and differentiability would be introduced through the compatibility equation in the elastic problems.

Since the shear deformations are ignored in our analysis of beam and plate problems, zero cross sections or zero thicknesses are found at the places where the bending moment vanishes. Shear stress would become infinite at the zero cross section. In order to make the design more realistic, shear deformations must be taken into account and the stress at any point must be restricted within the maximum allowable stress. However the analysis involving the shear deformation and the restriction of local stresses would become complicated.

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Абстракт—Целью настоящей работы является вывод общей теории оптимального расчета конструкции так, чтобы конструкция с заданым объемом обладала максимальной жесткостью. Выводится достаточное условие оптимальности из закона минимума потенциальной энергии, Это условие является необходимым в смысле вариационного метода тогда, как оптимальная конструкция обладает некоторыми свйоствами непреривности и дифференциальности. Обсуждаются физические значения условия оптимальности для задач балок, пластинок и ферм. Применение теории иллюстрируется задачей оптимального расчета свободно опертой, круглой пластинки под постоянном давлением. Дается детальное описание техники расчета для решения задачи пластинки.